Spectral Bandlimiting and Compact-Window Amenability

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Supplementary research note

This note supplements the-geometry-of-computable-trajectories.html, especially its spectral discussion, and the worked contrast in when-continuous-trajectories-can-and-cannot-compute.html. The geometry paper states a transfer-operator spectral criterion for membership in the amenable set $\mathcal{A}$; the worked examples paper shows, concretely, that bounded crossing density rather than smoothness is the decisive geometric feature. The present note gives a constructive Fourier-side surrogate: low-pass projection sends an L^2 signal into an entire-function class whose zero crossings are locally finite on every compact window.

This is intentionally weaker than the global function-space claim in the main geometry paper. It produces compact-window amenability rather than a uniform global dwell-time bound. That limitation matters, and it is stated explicitly below. The companion note finite-horizon-geometry-of-computational-amenability.md shows how such compact-window control feeds into a finite-horizon $(\varepsilon, \eta, \lambda)$ budget.

We work in the setting of the preceding papers. Let $\mathcal{A} \subset C(\mathbb{R})$ denote the set of trajectories admitting a coarse-graining Π with positive minimum dwell time $\tau_{\min} > 0$, and let $\mathcal{N} = C(\mathbb{R}) \setminus \mathcal{A}$ denote its complement. Since $\mathcal{A}$ is not a vector space, there is no orthogonal projection $C(\mathbb{R}) \to \mathcal{A}$. The Fourier transform nevertheless furnishes a canonical family of maps into an amenable subclass, parameterized by bandwidth.

Proposition (Spectral projection into a compact-window amenable class). Let $x \in L^2(\mathbb{R})$ have Fourier transform

$$\hat{X}(f) = \int_{\mathbb{R}} x(t) e^{-2\pi i f t}\, dt.$$

For $W > 0$, define the bandlimited projection

$$x_W(t) = \int_{-W}^{W} \hat{X}(f) e^{2\pi i f t}\, df.$$

Then:

(i) x_W extends to an entire function of exponential type $2\pi W$.

(ii) If $x_W \not\equiv 0$, then its zero set

$$Z(x_W) = \{t \in \mathbb{R} : x_W(t) = 0\}$$

is discrete and has no accumulation point. In particular, $Z(x_W) \cap [a, b]$ is finite for every compact interval $[a, b]$.

(iii) For the binary coarse-graining

$$\Pi(x) = \begin{cases} R, & x \geq 0, \\ L, & x < 0, \end{cases}$$

the minimum dwell time satisfies

$$\tau_{\min}(x_W, \Pi)\big|_{[a,b]} > 0$$

on every compact interval $[a, b]$. Consequently, x_W is computation-amenable on every compact observation window.

(iv) The number of zeros in any interval of length T satisfies

$$|Z(x_W) \cap [a, a+T]| \leq 2WT + O(1),$$

where the implicit constant depends on x_W but not on a or T.

Proof. Statement (i) is the Paley-Wiener theorem: a function whose Fourier transform has compact support in $[-W, W]$ extends to an entire function of exponential type $\sigma = 2\pi W$.

For (ii), the zeros of a nonzero entire function are isolated by the identity theorem. Since x_W is entire, if $x_W \not\equiv 0$, then $Z(x_W)$ has no accumulation point in $\mathbb{C}$, and hence none in $\mathbb{R}$. Finiteness on compact intervals follows immediately.

For (iii), finiteness of $Z(x_W) \cap [a, b]$ implies that the consecutive zeros $t_1 < \cdots < t_m$ have positive minimum gap,

$$\min_j (t_{j+1} - t_j) > 0.$$

Between consecutive zeros, continuity prevents x_W from changing sign, so the trajectory remains in a single cell of Π for at least that duration. Thus the induced symbolic trajectory has positive minimum dwell time on every compact interval.

For (iv), the classical zero-density estimate for entire functions of exponential type σ gives a counting-function bound

$$n(T) \leq \frac{\sigma T}{\pi} + O(1).$$

Substituting $\sigma = 2\pi W$ yields

$$n(T) \leq 2WT + O(1).$$

This completes the proof.

Definition (Spectral amenability ratio). Let $x \in L^2(\mathbb{R})$ with $\|x\|_2 > 0$, and let the dwell-time target be $\tau > 0$. Set

$$W(\tau) = \frac{1}{2\tau}$$

and define

$$\rho_{\mathcal{A}}(x, \tau) = \frac{\displaystyle\int_{-W(\tau)}^{W(\tau)} |\hat{X}(f)|^2 \, df}{\displaystyle\int_{\mathbb{R}} |\hat{X}(f)|^2 \, df}.$$

By Parseval's theorem,

$$\rho_{\mathcal{A}}(x, \tau) = \frac{\|x_{W(\tau)}\|_2^2}{\|x\|_2^2} \in [0, 1],$$

so $\rho_{\mathcal{A}}$ is the fraction of total signal energy preserved by the spectral projection into the compact-window amenable class at timescale τ. The complementary fraction,

$$1 - \rho_{\mathcal{A}}(x, \tau),$$

is the energy that must be discarded, or physically damped, to enforce amenability at that timescale.

Basic properties.

(a) $\rho_{\mathcal{A}}(x, \tau)$ is monotone decreasing in τ: demanding longer dwell times forces the projection to discard more high-frequency energy.

(b) For a pure tone $x(t) = \sin(\omega_0 t)$,

$$\rho_{\mathcal{A}}(x, \tau) = 1$$

whenever $\tau < \pi/\omega_0$, since all of the energy is concentrated at the two frequencies $\pm\omega_0/(2\pi)$.

(c) For a Brownian-like finite-window surrogate with broad spectral mass and an effective $1/f^2$ tail, one expects

$$\rho_{\mathcal{A}}(x, \tau) \to 0 \quad \text{as} \quad \tau \to \infty,$$

not as $\tau \to 0$. Long required dwell times force almost all of the energy in the high-frequency tail to be discarded. This matches the geometry paper's claim that generic Brownian paths are non-amenable, while also respecting the present note's narrower L^2 setup: actual Brownian paths are not themselves elements of $L^2(\mathbb{R})$.

This heuristic is the analytic companion to fig4_script.py, which plots the same monotone comparison for pure-tone, narrowband, Brownian-like, and white-noise signal classes.

Remark. As written, part (iii) proves compact-window amenability. A uniform global lower bound on dwell time would require an additional hypothesis guaranteeing uniform separation of real zeros.