How large must a triangle be to survive?

Minimum-distance exclusion under two sampling laws

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A triangle can have a long diameter and still have two vertices too close together to satisfy a minimum-distance requirement. Enlarging the triangle eventually resolves the pair. How large must a randomly drawn shape become? We compare two ways of drawing a triangle. One loses a greater fraction of shapes at every diameter above the cutoff. Its mean required diameter is infinite, although each individual shape has a finite threshold.

Shapes and scale. Take the three vertices as the required points and let $k > 0$ be the minimum permitted distance between distinct points. The diameter D is the longest side. If S is the ratio of shortest to longest side, the closest two vertices lie DS apart. The shape first satisfies the cutoff at diameter $M = k/S$, with equality allowed. Thus the fraction lost at a given diameter D is $\Pr(M > D) = \Pr(S < k/D)$.

Under law R, choose an acute angle θ uniformly in $(0, \pi/2)$ and form a right triangle with unit hypotenuse. Under law A, fix one side at one and draw the other two lengths (u, v) uniformly by area from $T = \{(u, v) \in (0, 1]^2 : u + v > 1\}$. The unit side is longest and the inequality ensures a nondegenerate triangle. Both laws fix the same diameter before rescaling.

Theorem. *For every $D > k$, the minimum surviving diameters obey $\Pr(M_R > D) > \Pr(M_A > D)$. Their means satisfy $\mathbb{E}[M_R] = \infty$ and $\mathbb{E}[M_A] = 4k \log 2$.*

Derivation of the exclusion probabilities. Write $t = k/D$ and $E_i(t) = \Pr(S_i < t)$ for $i = R, A$. The legs of the right triangle are $\sin\theta$ and $\cos\theta$. For $0 < t \le 1/\sqrt{2}$, both reach the cutoff exactly when $\arcsin t \le \theta \le \pi/2 - \arcsin t$. Dividing the excluded angular length by $\pi/2$ gives

$$E_R(t) = \begin{cases} (4/\pi)\arcsin t, & 0 < t \le 1/\sqrt{2}, \\ 1, & t > 1/\sqrt{2}. \end{cases}$$

At $t = 1/\sqrt{2}$ only the isosceles right triangle survives, a single angle of probability zero.

For law A, $S_A = \min(u, v)$ and T has area $1/2$. When $t \le 1/2$, the corner with $u < t$ has area $\int_0^t u\,du = t^2/2$, since $1 - u < v \le 1$. The corner with $v < t$ has the same area and the two are disjoint. For $1/2 < t < 1$, the surviving square $[t, 1]^2$ lies entirely in T. Dividing by the total area gives

$$E_A(t) = \begin{cases} 2t^2, & 0 < t \le 1/2, \\ 1 - 2(1-t)^2, & 1/2 < t < 1, \\ 1, & t \ge 1. \end{cases}$$

Ordering at every diameter. At $D = 2k$ the two curves give losses of $2/3$ and $1/2$, respectively. To see why their order persists, bound $E_A(t)$ by a straight line through the origin. The least possible slope is the largest value of

$$\frac{E_A(t)}{t} = \begin{cases} 2t, & 0 < t \le 1/2, \\ 4 - 2t - 1/t, & 1/2 < t \le 1. \end{cases}$$

The first branch increases to one. The second has derivative $-2 + 1/t^2$, positive below $1/\sqrt{2}$ and negative above. The maximum slope is therefore $4 - 2\sqrt{2} \simeq 1.172$, below $4/\pi \simeq 1.273$. Since $\arcsin t \ge t$, for $0 < t \le 1/\sqrt{2}$ we have

$$E_A(t) \le (4 - 2\sqrt{2})t < (4/\pi)t \le E_R(t).$$

For $1/\sqrt{2} < t < 1$, the right-triangle loss is already one, while the side-sampled law retains a square of positive area. Together these intervals cover every $D > k$. At $D = k$ only an equilateral triangle can pass, and its probability under law A is zero. The two loss probabilities then coincide.

Mean required diameter. The same narrow shapes that determine the large-D loss also determine whether M has a finite mean. For a right triangle, symmetry exchanges the two legs at $\theta = \pi/4$. On the lower half of the angular interval, $S_R = \sin\theta$, so

$$\frac{\mathbb{E}[M_R]}{k} = \frac{4}{\pi}\int_0^{\pi/4} \frac{d\theta}{\sin\theta} \;\ge\; \frac{4}{\pi}\int_0^{\pi/4} \frac{d\theta}{\theta} = \infty.$$

Under law A, differentiating the continuous, piecewise smooth function E_A gives density $4s$ for $0 < s < 1/2$ and $4(1 - s)$ for $1/2 < s < 1$. Consequently

$$\begin{aligned} \frac{\mathbb{E}[M_A]}{k} &= \int_0^{1/2} \frac{4s}{s}\, ds + \int_{1/2}^{1} \frac{4(1-s)}{s}\, ds \\ &= 2 + 4(\log 2 - 1/2) = 4\log 2. \end{aligned}$$

This proves both claims. □

The difference comes from the room each law gives to a short side. Under uniform angles, a leg shorter than t occupies an angular interval of order t at either endpoint. Under uniform side pairs, a side of length $u < t$ also confines its partner to an interval of width u. Integrating that width gives a loss of order t^2. At large diameters the excluded fractions thus decay as D^{-1} and D^{-2}, respectively. The choice of how to sample a shape determines whether its mean required diameter exists.

Related work. The angular calculation follows J. R. Landers, *Consequences of Minimum Distance in Mathematics and Physics*, section on surviving triangles. For side distributions obtained by sampling directions independently, see *Size distributions in random triangles*, *Journal of Applied Probability*.